Appendix E
Mathematics Intervention and Algebra Readiness Instructional Materials

Most students can be well served by basic instructional materials that include strategies to address a wide variety of instructional settings, but some may still experience intermittent difficulties that require focused intervention. A program of intervention or algebra readiness should:

- Include a balance of computational and procedural skills, conceptual understanding, and problem solving, as described in Chapter 1 in this framework.
- Prioritize the concepts and skills to be taught so that the teacher can make optimal use of time and resources and provide an adequate sampling of the range of examples that define each concept. These instructional examples should be unambiguous and presented in a logical sequence, moving from the very simple to the more complicated and from the concrete to the abstract.
- Provide clear goals and extensive diagnostic tools to assess students' mathematical knowledge. The entry-level assessments should identify which students need the program and their strengths and weaknesses.
- Provide suggestions for how the teacher can monitor student performance daily so that student confusion does not go undetected.
- Provide valid and reliable periodic assessments that can give ongoing information on the causes of student errors and misconceptions and advice for the types of interventions that can be used for each area of difficulty.
- Provide engaging, motivating materials that help students focus on the goals of the program.
- Provide tasks that require students to show their mathematical reasoning and problem-solving strategies so that the teacher can identify sources of students' incomplete or erroneous understanding of the underlying mathematics.
- Reflect the interests and the ages of the students (e.g., materials used to teach a foundational skill or concept to students in grade eight should reflect the interests of a teenager).

Overcoming student learning problems in mathematics requires giving attention to the background of the individual students and to the nature of their previous instruction. As reviewed by Chapman (1988), some students who need remediation perceive their low abilities to be unchangeable, expect to fail in the future, and give up readily when confronted with difficult tasks. Their continued failure confirms their low expectations of achievement, a pattern that perpetuates a vicious cycle of additional failure. What are needed are instructional programs that create steady measurable progress for students, showing them that whatever difficulties they might have had in the past, they are learning mathematics now.

Providing too many instructional directions for any student, with a loss of continuity in instruction, could be as bad as using too few. The goal is for the teacher to have a big instructional "toolbox" at hand from which to select exactly the tools needed for the class. These tools should adhere to the guidance in this appendix and in Chapter 10 and should be based on research.

Particular attention should be given to the needs of English learners, including the academic language of instruction and the specialized vocabulary of mathematics. If students do not understand the academic language of instruction and assessment, they will not succeed in mathematics. These areas should be addressed:

- At an early stage students may have difficulty with such English words as *first, second, last, before, every, more,* and *equal.* Students may be unfamiliar with *numerator, denominator, commutative,* and *equivalence* or may not understand a fraction decoded into words (e.g., *three halves*).
- The distinction between words that sound or are spelled the same, such as *tens* and *tenths,* is sometimes not noted or understood.
- The different meanings of multiple-meaning words should be explicitly taught. These words may have a meaning in common discourse that is different from the meaning in mathematics, such as *variable, function, plane, table,* or *draw* (as in *to draw a triangle* compared with *to draw a conclusion*).
- A related language issue is that the place values of some of the numbers between 10 and 20 are not obvious from their names (e.g., the number *16* is called *sixteen* in English, but *ten plus six* in other languages).
- Understanding narrative descriptions of a word problem can require language skills that students have not yet mastered, particularly when the language of a word problem is ambiguous or idiomatic.
- Materials should include opportunities to reinforce the specialized vocabulary of mathematics throughout the year.

The language of mathematics is very precise compared with the English used in common discourse, and this difference separates mathematics from most other curricular areas. Mathematical reasoning involves the use of logic in a system of precisely defined environments (e.g., the set of all whole numbers), concepts (e.g., addition, multiplication), and rules (e.g., the associative rule). Mathematical reasoning, as explained in the sections that follow and in the introductory section of Chapter 3, must be systematically embedded in the teaching of the subset of standards for mathematics intervention and algebra readiness programs.

This appendix provides two types of guidance for the development of mathematics intervention and algebra readiness materials addressing the specific standards identified for each type of program:

1. A discussion of the types of errors students are likely to make
2. An explanation of why learning these particular standards is important

Publishers are to use Appendix E in conjunction with Chapter 10 to design instructional materials.

In a survey of local educational agencies (LEAs) conducted during the spring of 2004, a curriculum and instruction steering committee of the California County Superintendents Educational Services Association collected information on how mathematics intervention programs are organized and implemented in California. Some LEAs reported that they conducted intervention programs before or after school hours or during an intersession or a summer session. Many schools provide additional instructional time for mathematics during regular school hours or conduct intervention in a tutorial setting. The 60 districts that responded to the survey were geographically distributed among 14 counties. The results of the survey indicate a need to provide materials that can be used in a variety of instructional settings.

The subsets of mathematics content standards for algebra readiness and mathematics intervention materials are shown in the chart that follows, "The Subsets of Mathematics Content Standards, by Grade Level."

The Subsets of Mathematics Content Standards, by Grade Level

Mathematics Standards for the Algebra Readiness Program

Grade 7	NS	1.2, 1.3, 1.5, 2.1
	AF	1.1, 1.3, 2.1, 3.3, 3.4 , 4.1, 4.2
	MG	1.3, 3.3
	MR	1.0, 1.1, 1.2, 1.3, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 3.0, 3.1, 3.2, 3.3
Algebra I		2.0, 4.0, 5.0
Grade 2	AF	1.1
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 3.0
Grade 3	NS	1.3, 1.5
	AF	1.5
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 4	NS	3.1, 3.2
	AF	1.2, 1.3, 1.5, 2.0, 2.1, 2.2
	MG	2.0, 2.1, 2.2, 2.3
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 5	NS	1.4
	AF	1.0, 1.3, 1.4
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 6	NS	1.1, 1.4, 2.0, 2.1, 2.2
	AF	1.0, 1.1
	MR	1.0, 1.1, 1.2, 1.3, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 3.0, 3.1, 3.2, 3.3

The Subsets of Mathematics Content Standards, by Grade Level (Continued)

Mathematics Standards for the Mathematics Intervention Program

Kindergarten	AF	1.1
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2
Grade 1	NS	1.1, 1.2, 1.3, 1.4, 2.1, 2.5, 2.6, 2.7
	SDAP	1.1, 1.2, 2.1
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 3.0
Grade 2	NS	1.1, 1.2, 1.3, 2.2, 2.3, 3.1, 3.3, 4.0, 4.1, 4.3, 5.1, 5.2
	AF	1.1
	MG	1.3
	SDAP	1.1, 1.2, 2.1
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 3.0
Grade 3	NS	1.3, 1.5, 2.1, 2.2, 2.4, 2.6, 2.7, 3.1, 3.2, 3.4
	AF	1.0, 1.4, 1.5, 2.1, 2.2
	MG	1.2, 1.3, 1.4
	SDAP	1.3
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 4	NS	1.1, 1.2, 1.3, 1.5, 1.6, 1.7, 1.8, 2.0, 3.1, 3.2, 4.1
	AF	1.1, 1.5, 2.1, 2.2
	MG	1.1, 2.0, 2.1, 2.2, 2.3
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 5	NS	1.2, 1.5, 2.0, 2.1, 2.5
	AF	1.2, 1.3, 1.5
	MG	1.1, 1.2, 1.3, 2.1, 2.2
	SDAP	1.3, 1.4, 1.5
	MR	1.0, 1.1, 1.2, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 3.0, 3.1, 3.2, 3.3
Grade 6	NS	1.1, 1.2, 1.3, 1.4, 2.1, 2.3
	AF	1.2, 2.1, 2.2, 2.3
	MG	1.2, 1.3, 2.2
	SDAP	3.3
	MR	1.0, 1.1, 1.2, 1.3, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 3.0, 3.1, 3.2, 3.3
Grade 7	NS	1.2, 1.3, 1.6, 1.7
	AF	1.1, 1.2, 1.3, 3.0, 3.1, 3.3, 3.4, 4.0, 4.2
	MG	1.1, 1.3, 3.3, 3.4
	MR	1.0, 1.1, 1.2, 1.3, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 3.0, 3.1, 3.2, 3.3

Abbreviations for the strands of the mathematics content standards are NS, Number Sense; AF, Algebra and Functions; MG, Measurement and Geometry; SDAP, Statistics, Data Analysis, and Probability; and MR, Mathematical Reasoning.

Note that the strand of Mathematical Reasoning is different from the other four strands. This strand, which is inherently embedded in each of the other strands, is fundamental in developing the basic skills and conceptual understanding for a solid mathematical foundation. It is important when looking at the standards to see the reasoning in all of them. Although the standards in the Mathematical Reasoning strand are not explicitly mentioned in the narrative for the intervention program or algebra readiness program, the Mathematical Reasoning standards must be systematically embedded in the teaching of the subsets of standards shown in this chart.

A Mathematics Intervention Program
(Grades Four Through Seven)

The six volumes described in this section represent the subset of standards that must be addressed in this highly focused program. It is designed to serve strategic and intensive students in grades four through seven so that they can learn efficiently from basic grade-level instructional materials. The program is not intended to serve as a fixed-term course and should not be used for tracking students. The embedded assessments should provide a plan for each student that identifies which sections of the six volumes need to be covered and when students are ready to move on to the next section or to exit the program.

A critical design element of the intervention instructional materials for grades four through seven is that the materials should focus on the subset of standards described in this section and should break each standard into a series of small conceptual steps and embedded skills.

Materials must also be organized around the six volumes and their indicated standards. No specific order of the topics within these volumes is required, and volumes may be split into smaller units for publication. The intent of this organization is to support the flexible use of intervention materials at the school site.

Six Volumes for the Mathematics Intervention Program

Volume I **Place Value and Basic Number Skills**	**Grade 1**	NS	1.1, 1.2, 1.3, 1.4, 2.1, 2.5, 2.6, 2.7
	Grade 2	NS	1.1, 1.2, 1.3, 2.2, 2.3, 3.1, 3.3
	Grade 3	NS	1.3, 1.5, 2.1, 2.2, 2.4, 2.6
	Grade 4	NS	1.1, 1.2, 1.3, 1.6, 3.1, 3.2, 4.1
Volume II **Fractions and Decimals**	**Grade 2**	NS	4.0, 4.1, 4.3, 5.1, 5.2
	Grade 3	NS	3.1, 3.2, 3.4
	Grade 4	NS	1.5, 1.6, 1.7, 1.8, 2.0
	Grade 5	NS	1.5, 2.0, 2.1, 2.5
	Grade 6	NS	1.1, 2.1, 2.3
	Grade 7	NS	1.2
Volume III **Ratios, Rates, and Percents**	**Grade 3**	NS	2.7
		AF	1.4, 2.1, 2.2
		MG	1.4
	Grade 5	NS	1.2
		SDAP	1.3
	Grade 6	NS	1.2, 1.3, 1.4
		AF	2.1, 2.2, 2.3
		SDAP	3.3
	Grade 7	NS	1.6, 1.7
		AF	4.2

Volume IV **The Core Processes of Mathematics**	**Grade 2**	AF	1.1
	Grade 3	AF	1.0, 1.5
	Grade 4	AF	1.1, 2.1, 2.2
	Grade 5	AF	1.2, 1.3
	Grade 6	AF	1.2
	Grade 7	NS	1.3
		AF	1.1, 1.2, 1.3, 4.0
Volume V **Functions and Equations**	**Kindergarten**	AF	1.1
	Grade 1	SDAP	1.1, 1.2, 2.1
	Grade 2	SDAP	1.1, 1.2, 2.1
	Grade 3	AF	2.1, 2.2
		SDAP	1.3
	Grade 4	AF	1.5
		MG	2.0, 2.1
	Grade 5	AF	1.5
		SDAP	1.4, 1.5
	Grade 6	NS	1.3
		AF	2.1
	Grade 7	AF	3.0, 3.1, 3.3, 3.4
Volume VI **Measurement**	**Grade 2**	MG	1.3
	Grade 3	AF	1.4
		MG	1.2, 1.3, 1.4
	Grade 4	MG	1.1, 2.2, 2.3
	Grade 5	MG	1.1, 1.2, 1.3, 2.1, 2.2
	Grade 6	AF	2.1
		MG	1.2, 1.3, 2.2
	Grade 7	MG	1.1, 1.3, 3.3, 3.4

Abbreviations for the strands of the mathematics content standards are NS, Number Sense; AF, Algebra and Functions; MG, Measurement and Geometry; SDAP, Statistics, Data Analysis, and Probability; and MR, Mathematical Reasoning. **The standards from the Mathematical Reasoning strand are not displayed in this chart (see pp. 340–41 for the MR standards).**

Volume I Place Value and Basic Number Skills

This volume is about place value and basic number skills and covers the following topics: I-1. Counting; I-2. Place Value; I-3. Addition and Subtraction; I-4. Multiplication; and I-5. Division.

I-1. Counting

Number Sense (Grade One)

1.1 Count, read, and write whole numbers to 100.

1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than (<, =, >).

1.3 Represent equivalent forms of the same number through the use of physical models, diagrams, and number expressions (to 20) (e.g., 8 may be represented as 4 + 4, 5 + 3, 2 + 2 + 2 + 2, 10 – 2, 11 – 3).

1.4 Count and group objects in ones and tens (e.g., three groups of 10 and 4 equals 34, or 30 + 4).

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).

Number Sense (Grade Two)

1.3 Order and compare whole numbers to 1,000 by using the symbols <, =, >.

3.1 Use repeated addition, arrays, and counting by multiples to do multiplication.

3.3 Know the multiplication tables of 2s, 5s, and 10s (to “times 10”) and commit them to memory.

This section is about the arithmetic of whole numbers. Beginning students may be able to read numerals before they possess the skills of rote counting (naming numbers 1, 2, 3, . .) or of rational counting (assigning one and only one number name to each object in a group, knowing where to start and stop counting the objects, and knowing when all have been counted correctly). They may not realize that the ordering of numbers is related to how they are counted (e.g., 37 > 23 because 37 comes after 23). At an early stage students may be helped by concrete or pictorial or graphic representations of numbers (e.g., numbers on a number line, scaled area models of 1, 10, 100, 1,000) and of operations (e.g., multiplication visualized with an area model or with groups of discrete objects).

Numbers beyond nine are represented by creating a tens place (to the left) and beyond 99 by creating a hundreds place. Each new place has a value ten times that of the place immediately to the right. The number 1,000, for example, is ten steps from zero if counting is performed by 100s. This foundational work on numbers will help students understand place value (e.g., the complete expanded form of numbers), which will help them understand the addition and subtraction algorithms included in topic I-3. "Addition and Subtraction."

I-2. Place Value

Number Sense (Grade Two)

1.1 Count, read, and write whole numbers to 1,000 and identify the place value for each digit.

1.2 Use words, models, and expanded forms (e.g., 45 = 4 tens + 5) to represent numbers (to 1,000).

Number Sense (Grade Three)

1.3 Identify the place value for each digit in numbers to 10,000.

1.5 Use expanded notation to represent numbers (e.g., 3,206 = 3,000 + 200 + 6).

Number Sense (Grade Four)

1.1 Read and write whole numbers in the millions.

1.2 Order and compare whole numbers and decimals to two decimal places.

1.3 Round whole numbers through the millions to the nearest ten, hundred, thousand, ten thousand, or hundred thousand.

1.6 Write tenths and hundredths in decimal and fraction notations and know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5$ or 0.50; $\frac{7}{4} = 1\frac{3}{4} = 1.75$).

Students should know that the zero in a number such as 3,206 has meaning as a coefficient (3 thousands + 2 hundreds + 0 tens + 6 ones), and they may become confused if they think that zero is a placeholder for a number rather than a number itself. Students need to understand rounding as an issue of place value (e.g., 397 rounded to the nearest ten is 40 tens, and 207 rounded to the nearest ten is 21 tens). Number Sense Standard 1.6 (grade four) is intended as an advanced subtopic because students may have difficulty in understanding the numerical relationships.

Place value plays a critical role in all the arithmetic algorithms of whole numbers (see the example in "Algebra Readiness [Grade Eight or Above]," Topic 2. "Operations on Whole Numbers"). The concept that adjacent place value columns are related by multiplication or division by ten may help students understand standard algorithms (e.g., regrouping during an operation) and evaluate the reasonableness of calculated results.

I-3. Addition and Subtraction

Number Sense (Grade One)

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.6 Solve addition and subtraction problems with one- and two-digit numbers (e.g., 5 + 58 = __).

2.7 Find the sum of three one-digit numbers.

Number Sense (Grade Two)

2.2 Find the sum or difference of two whole numbers up to three digits long.

2.3 Use mental arithmetic to find the sum or difference of two two-digit numbers.

Number Sense (Grade Three)

2.1 Find the sum or difference of two whole numbers between 0 and 10,000.

Number Sense (Grade Four)

3.1 Demonstrate an understanding of, and the ability to use, standard algorithms for the addition and subtraction of multidigit numbers.

Although this statement is a simplification, addition of whole numbers is "counting on" in the sense that 12 + 5 is the number a student arrives at by counting by ones 5 more times, starting after 12. From a practical perspective the algorithms for the addition and subtraction of whole numbers are important because they replace the need to count forward and backward by ones to determine sums and differences. The standard algorithms are applications of the definitions of mathematical operations to numbers in place value notation. The use of algorithms is not always the most efficient method of problem solving, but they are sufficiently robust that they *can* be applied in every case. The standard algorithms model the mathematical strategy of breaking complicated problems into smaller solvable components and help to develop later algebraic concepts of working with polynomials.

Understanding place value and such related processes as regrouping (or carrying) of multidigit numbers is important, as are techniques of mental arithmetic. Students need to understand how familiar algorithms work, recognize the different situations in which operations are called for strategically in problem solving, and have sufficient practice with different types of problems so that they can generalize and apply that knowledge to new and novel situations.

At an early stage students may have difficulties if they do not possess the skills of counting and reading numerals or understand the concept of equality (e.g., in problems that have a missing addend, such as 5 + __ = 7). As a symbol, the meaning of the equals sign might be misinterpreted as "and the answer is" or "compute this," which are the functions of the equal sign on a calculator, instead of as a mathematical symbol representing a statement about whole numbers that when numbers on both sides of the symbol are counted, they are the same. (Later, students will see equality between numbers as a symmetric and transitive relationship.)

I-4. Multiplication

Number Sense (Grade Three)

2.2 Memorize to automaticity the multiplication table for numbers between 1 and 10.

2.4 Solve simple problems involving multiplication of multidigit numbers by one-digit numbers ($3{,}671 \times 3 = \_\_$).

2.6 Understand the special properties of 0 and 1 in multiplication and division.

Number Sense (Grade Four)

4.1 Understand that many whole numbers break down in different ways (e.g., $12 = 4 \times 3 = 2 \times 6 = 2 \times 2 \times 3$).

Multiplication is shorthand for repeated addition when we count groups of the same size. In the previous illustration of 1,000 being ten steps from zero when we count by 100s, this idea is expressed as 1,000 = 10 x 100. The language "three times four" means the number of objects in "three sets of four" objects, or "three fours."

At an early stage students can conceptualize multiplication by using concrete examples (e.g., counting sets of the same size and counting by 2s, 5s, 10s or the representation of equivalent sets of discrete objects). Such models may help students conceptualize multiplication by zero, using mathematical reasoning and the definition of multiplication (e.g., zero added to itself five times is still zero; therefore, $5 \times 0 = 0$). The special case of multiplication by 1 can also be similarly conceptualized.

Students may have trouble if they do not see the logical patterns and order in the multiplication table or do not commit this table to memory. Practice with a wide variety of problems is important because if the students have solved only regular problems of the type $3 \times 4 = \_\_$, then a problem that has a missing factor, such as $3 \times \_\_ = 12$, might be confusing to them and elicit the incorrect answer $3 \times \underline{36} = 12$.

The decomposition of numbers, for example, by factorization, and the recombination of composite numbers are important for introducing students to the idea that the same number can have different representations.

I-5. Division

Number Sense (Grade Four)

3.2 Demonstrate an understanding of, and the ability to use, standard algorithms for multiplying a multidigit number by a two-digit number and for dividing a multidigit number by a one-digit number; use relationships between them to simplify computations and to check results.

The connections between multiplication and division must be made evident to students early. A division statement such as $8 \div 4 = 2$ is simply an alternative and equivalent way of saying $8 = 2 \times 4$, so a problem such as $8 \div 4 = \_\_$ is equivalent to a problem that has a missing factor $8 = \_\_ \times 4$. Such a clear-cut explanation of division would help students understand why division by 0 cannot be defined (e.g., $5 \div 0$ does not make sense because there is no numerical answer to the problem that has a missing factor $5 = 0 \times \_\_$), why division by 1 does not change a number, and why division of whole numbers does not always yield a whole number (e.g., $2 \div 7$). It follows that $8 \div 4 = 2$ has the intuitive meaning of partitioning 8 objects into equal groups of 4 objects and that there are 2 such groups, or 4 groups with 2 objects in each group.

Students may have difficulty early if they have not seen a wide variety of demonstrations of division in which a group of concrete objects is divided into smaller groups of equal size (with no remainder). Lingering difficulties with multiplication facts or subtraction of multidigit numbers may impede students' success with long division, so it is important that instruction be highly focused.

Volume II Fractions and Decimals

This volume is about fractions and decimals and covers the following topics: II-1. Parts of a Whole; II-2. Equivalence of Fractions; II-3. Operations on Fractions; II-4. Decimal Operations; and II-5. Positive and Negative Fractions and Decimals.

II-1. Parts of a Whole

Number Sense (Grade Two)

4.0 Students understand that fractions and decimals may refer to parts of a set and parts of a whole.

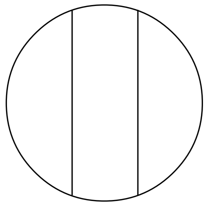
4.1 Recognize, name, and compare unit fractions from $\frac{1}{12}$ to $\frac{1}{2}$.

4.3 Know that when all fractional parts are included, such as four-fourths, the result is equal to the whole and to one.

5.1 Solve problems using combinations of coins and bills.

5.2 Know and use the decimal notation and the dollar and cent symbols for money.

Common models for fractions are the partition or decomposition of a set (e.g., one dollar partitioned into four quarters) or of an area (e.g., a rectangle divided into four parts of equal area) and the points on a number line. However, for fractions the teacher needs to specify "the whole" explicitly before discussing "parts" of the whole. For example, if the teacher uses subsets of circles to illustrate fractions, then he or she should specify clearly that "the whole" is the AREA of a circle, so that "the parts" will be the areas of some of the circle's subsets. Otherwise, students may divide a circle into three subsets of equal width and claim that each subset is $\frac{1}{3}$, drawing figure 1 instead of figure 2, as shown in the illustrations that follow:



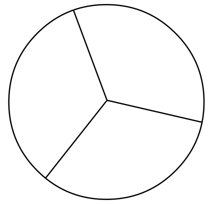


Figure 1 instead of **Figure 2**

In the early grades representations of fractions as concrete objects are important as a conceptual foundation so that students can make the transition later to a more precise and generalized definition of rational numbers and their operations. The ability to use different representations of fractions is important as a foundation for later work in algebra.

II-2. Equivalence of Fractions

Number Sense (Grade Three)

3.1 Compare fractions represented by drawings or concrete materials to show equivalency and to add and subtract simple fractions in context (e.g., $\frac{1}{2}$ of a pizza is the same amount as $\frac{2}{4}$ of another pizza that is the same size; show that $\frac{3}{8}$ is larger than $\frac{1}{4}$).

3.2 Add and subtract simple fractions (e.g., determine that $\frac{1}{8}+\frac{3}{8}$ is the same as $\frac{1}{2}$).

Number Sense (Grade Four)

1.5 Explain different interpretations of fractions, for example, parts of a whole, parts of a set, and division of whole numbers by whole numbers; explain equivalence of fractions (see Standard 4.0).

Number Sense (Grade Five)

1.5 Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers.

Number Sense (Grade Six)

1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

The equivalence of fractions and the addition of fractions are two substantive concepts that need to be developed carefully. Without a clear definition of what addition of fractions means, a beginning student might incorrectly think, on the basis of addition of "wholes" and "parts" of rectangular areas, that

$$\frac{2}{3}+\frac{2}{3}=\frac{(2+2)}{(3+3)}=\frac{4}{6},$$

or the student might argue that $\frac{1}{8}>\frac{1}{6}$ because $8 > 6$. Without a precise meaning for fractions (such as, for example, a point on the number line obtained in a prescribed way), students would have difficulty understanding equivalence of fractions and operations on fractions.

Familiarity with common benchmark fractions, such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and so forth, and the ability to estimate relative sizes of fractions (e.g., knowing that $\frac{7}{16}$ is almost $\frac{1}{2}$) may help students work with number lines.

The student should understand that two fractions (or any numbers for that matter) are equal when they are located at the same point on a number line, and if they are not equal, are ordered by their relative positions on the line. It is important from the start to establish the unit underlying the fraction. Starting with a unit, the teacher divides the unit into b equal parts and takes one or more of them, calling that number of parts a. A fraction $\frac{a}{b}$ can then be understood visually as the area of a single part or segment when the whole area of the unit is partitioned into b parts of equal area, as shown in figure 3, the visual model of fraction $\frac{a}{b}$.

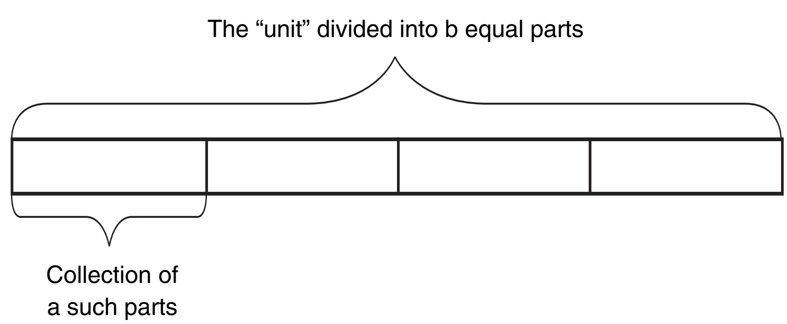


Figure 3

Addition of fractions with the same denominator can be readily seen as the addition of lengths on the number line. An understanding of equivalence is a prerequisite for the addition of fractions with unequal denominators because once students see that $\frac{a}{b}$ and $\frac{ad}{bd}$ are equivalent (because $\frac{d}{d} = 1$; a fraction is 1 when its numerator equals its denominator), and that $\frac{c}{d}$ and $\frac{cb}{bd}$ are equivalent, then $\frac{a}{b} + \frac{c}{d}$ is the same as $\frac{ad}{bd} + \frac{cb}{bd}$, which is a problem they already know how to model on the number line and calculate symbolically.

A mixed number, such as $3\frac{1}{2}$, must be carefully explained as $3 + \frac{1}{2}$. From this understanding, students can reason that this number is $\frac{3}{1} + \frac{1}{2}$, which is $\frac{6}{2} + \frac{1}{2}$, which by the meaning of addition of fractions with a common denominator is $\frac{(6+1)}{2}$, or $\frac{7}{2}$. These logical underpinnings allow students to justify calculations by using mathematical reasoning rather than unsubstantiated formulas.

II-3. Operations on Fractions

Number Sense (Grade Five)

2.5 Compute and perform simple multiplication and division of fractions and apply these procedures to solving problems.

Number Sense (Grade Six)

2.1 Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.

Students need to develop facility with moving among different representations of rational numbers and to understand what the operations mean. Working with fractions in a structured environment, with both careful definitions and application of the concepts to problem solving, serves as a foundation for future work with complex algebraic expressions. Liping Ma (1999, 55–83) has published examples of Chinese and U.S. teachers' contextual models for the division problem $1\frac{3}{4} \div \frac{1}{2}$; however, bringing real-world meaning to the division of fractions is difficult because many such models cannot be generalized to explain the reciprocal division problem $\frac{1}{2} \div 1\frac{3}{4}$.

For this reason, a precise definition of division is needed. Students first need to understand the precise meaning of multiplication of fractions that is consistent with multiplication of whole numbers (e.g., $\frac{a}{b} \times \frac{c}{d}$ = the area of a rectangle having sides of length $\frac{a}{b}$ and $\frac{c}{d}$). The meaning of the division of fractions is consistent with the meaning discussed earlier for whole numbers (see I-5. "Division"); namely, that $\frac{a}{b} \div \frac{c}{d}$ has the same solution as a problem that has a missing factor $\frac{a}{b} = \_\_ \times \frac{c}{d}$. The student can see that the "invert and multiply" algorithm for division $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ is based on mathematical reasoning because the product on the right, $\frac{ad}{bc}$, is the answer to the problem that has a missing factor $\frac{a}{b} = \_\_ \times \frac{c}{d}$. (The following steps verify logically how the value $\frac{ad}{bc}$ satisfies the equation; that is, results in the value $\frac{a}{b}$ for the right side of the equation: $\frac{ad}{bc} \times \frac{c}{d} = \frac{adc}{bdc} = \frac{a}{b} \times 1 = \frac{a}{b}$.)

II-4. Decimal Operations

Number Sense (Grade Four)

2.0 Students extend their use and understanding of whole numbers to the addition and subtraction of simple decimals.

Number Sense (Grade Five)

2.0 Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals.

Number Sense (Grade Six)

1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on the number line.

Fractions with a power of 10 as the denominator are examples of finite or terminating decimals. Students should know that the place value notation for whole numbers can be extended to decimals. Fractions with denominators of 10 and 100 are especially important in logically explaining decimal representation of currency and in moving between different equivalent representations (e.g., $5\frac{1}{2} = 5\frac{5}{10} = 5.5 = 5\frac{50}{100} = 5.50$).

Students should understand why in an addition problem, columns are lined up by the decimal points of the numbers rather than by the right-most digits.

A knowledge of place value concepts is basic to a student's ability to round off decimals (for example, knowing that $\frac{2}{3}$ is 0.67 rounded to the nearest hundredths) and to multiply decimals, with a reasonable estimate of the result prior to calculation.

II-5. Positive and Negative Fractions and Decimals

Number Sense (Grade Two)

5.2 Know and use the decimal notation and the dollar and cent symbols for money.

Number Sense (Grade Three)

3.4 Know and understand that fractions and decimals are two different representations of the same concept (e.g., 50 cents is $\frac{1}{2}$ of a dollar, 75 cents is $\frac{3}{4}$ of a dollar).

Number Sense (Grade Four)

1.6 Write tenths and hundredths in decimal and fraction notations and know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5$ or 0.50; $\frac{7}{4} = 1\frac{3}{4} = 1.75$).

1.7 Write the fraction represented by a drawing of parts of a figure; represent a given fraction by using drawings; and relate a fraction to a simple decimal on a number line.

1.8 Use concepts of negative numbers (e.g., on a number line, in counting, in temperature, in "owing").

Number Sense (Grade Five)

2.1 Add, subtract, multiply, and divide with decimals; add with negative integers; subtract positive integers from negative integers; and verify the reasonableness of the results.

Number Sense (Grade Six)

2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.

Number Sense (Grade Seven)

1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

Students may have seen the rational number 3.5 represented as a fraction ($\frac{7}{2}$), as an equivalent fraction ($\frac{35}{10}$), as an equivalent decimal (3.50), or visually as an area or a point on a number line. Students may need many opportunities to express a fraction as an equivalent decimal and as an equivalent fraction. Problems based on concrete situations and placed in context may be helpful.

Negative fractions and decimals may be developed from the same conceptual idea as negative integers (e.g., a problem that has a missing addend, such as

$\frac{5}{2} + \_\_ = 0$, is analogous to $5 + \_\_ = 0$). They can also be developed by first placing them on the number line so that for any fraction $\frac{a}{b}$, $(-\frac{a}{b})$ is the point symmetric to $\frac{a}{b}$ with respect to 0. Once students understand where negative fractions are located on the number line, they may more easily see that subtracting $\frac{5}{2}$ from $\frac{7}{2}$ is the same as adding $(-\frac{5}{2})$ to $\frac{7}{2}$, or $\frac{7}{2} - \frac{5}{2} = \frac{7}{2} + \left(-\frac{5}{2}\right)$.

Volume III Ratios, Rates, and Percents

This volume is about place value and basic number skills and covers the following topics: III-1. Ratio and Unit Conversion; III-2. Proportion and Percent; and III-3. Rates.

III-1. Ratio and Unit Conversion

Number Sense (Grade Three)

2.7 Determine the unit cost when given the total cost and number of units.

Algebra and Functions (Grade Three)

1.4 Express simple unit conversions in symbolic form (e.g., __ inches = __ feet × 12).

Measurement and Geometry (Grade Three)

1.4 Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).

Topic III-1 introduces simple ideas of relationships between numbers, but that topic may be difficult for students if they lack grounding in and precise definitions of the operations of multiplication and division. Understanding proportional reasoning in concrete situations may be difficult if a student has not had sufficient practical experience with common measures; for example, knowing whether a kilogram or a gram is the greater mass. Unit conversions are useful for helping students understand proportional reasoning and linear relationships. Unit conversions are also useful for scaling problems. Students should understand the equivalence of expressions in different units (e.g., 12 inches = 1 foot) and ultimately develop facility in moving between different representations of numbers (e.g., the "ratio of *a* to *b*" is the same as the quotient $\frac{a}{b}$; a ratio of 42 miles driven for every gallon of gas can be expressed as the quotient $\frac{42 \text{ miles}}{1 \text{ gallon}}$). Unit conversions are important because they help to facilitate mathematical understanding and because they frequently appear in practical applications.

III-2. Proportion and Percent

Algebra and Functions (Grade Three)

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Number Sense (Grade Five)

1.2 Interpret percents as a part of a hundred; find decimal and percent equivalents for common fractions and explain why they represent the same value; compute a given percent of a whole number.

Statistics, Data Analysis, and Probability (Grade Five)

1.3 Use fractions and percentages to compare data sets of different sizes.

Number Sense (Grade Six)

1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations ($\frac{a}{b}$, a to b, a:b).

1.3 Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7} = \frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

1.4 Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

Students need exposure to many types of problems in proportion and percent. In this topic students add percent to their repertoire as a special type of ratio in which the denominator is normalized to 100. Here are some examples: *A school with 100 students has 55 girls. What percentage of the students are girls? A school has 55 girls and 45 boys. What percentage of the students are girls? A worker earning $10 per hour was given a 10% increase in salary one year, then a 10% decrease in salary the next. What is the worker's salary after these two changes?*

Extending linear patterns and making generalizations about "what always happens" in a proportional relationship are important as students develop a more algebraic view of arithmetic.

III-3. Rates

Algebra and Functions (Grade Six)

2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

2.2 Demonstrate an understanding that rate is a measure of one quantity per unit value of another quantity.

2.3 Solve problems involving rates, average speed, distance, and time.

Statistics, Data Analysis, and Probability (Grade Six)

3.3 Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, 1-P is the probability of an event not occurring.

Algebra and Functions (Grade Seven)

4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Number Sense (Grade Seven)

1.6 Calculate the percentage of increases and decreases of a quantity.

1.7 Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

Students need to learn to operate with complex fractions. The concept of a fraction can be generalized so that both the numerator and denominator, instead of being restricted only to whole numbers, may themselves also be fractions (including decimals). The complex fraction $(\frac{a}{b})/(\frac{c}{d})$, where a, b, c, and d are whole numbers (or integers) is defined as the fraction $\frac{a}{b}$ divided by $\frac{c}{d}$. With this definition of a complex fraction, the usual formulas for adding, subtracting, multiplying, and dividing fractions generalize without change, and when the numerator and denominator of a complex fraction are each multiplied by the same rational number, the result is a fraction equal to the original. These important facts can be deduced directly from the previous definition of a complex fraction.

An application of complex fractions involves units of measure. Complex fractions arise naturally as percentages and in practical problems involving different units of measure. Complex fractions also lay the groundwork for understanding ratios of real numbers, for defining slopes of straight lines in beginning algebra, and even for understanding the concept of derivative in calculus.

Students may have been exposed to fractions as parts of a whole, but it is equally important that they are aware of the interpretation of a fraction as a ratio, in the sense that the fraction is used as a multiplicative comparison of the numerator with the denominator. Students should also understand that a fraction can be the ratio of one quantity to another, even if the quantities are not whole numbers.

Volume IV The Core Processes of Mathematics

This volume is about the core processes of mathematics and covers the following topics: IV-1. The Use of Symbols; IV-2. Mathematical Fundamentals; IV-3. Evaluating Expressions; IV-4. Equations and Inequalities; and IV-5. Symbolic Computation.

IV-1. The Use of Symbols

Algebra and Functions (Grade Three)

1.0 Students select appropriate symbols, operations, and properties to represent, describe, simplify, and solve simple number relationships.

Algebra and Functions (Grade Four)

1.1 Use letters, boxes, or other symbols to stand for any number in simple expressions or equations (e.g., demonstrate an understanding and the use of the concept of a variable).

Algebra and Functions (Grade Five)

1.2 Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

Algebra and Functions (Grade Six)

1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions (Grade Seven)

1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).

Topic IV-1 relates to the transition from a narrative description of a situation to a symbolic representation. To master the basics of algebra, students need to be fluent in processing the meaning of word problems and in translating that information into mathematical expressions and equations. Assigning a symbol, usually called a *variable,* to represent an unknown part of the expression is crucial to this process. For beginning students to comprehend the concept, the teacher must specify the symbols in a problem (e.g., *Find the number x so that* $27 - x = 14$). Symbolic manipulation should not be memorized by students as a series of tricks (e.g., FOIL for binomial multiplication) but should be understood through mathematical reasoning (e.g., application of the distributive property). This ability will allow students to have insight into the mathematics and be able to solve types of problems they might not have seen; for example, expansion of $(a + b)(x + y + z)$.

IV-2. Mathematical Fundamentals

Algebra and Functions (Grade Two)

1.1 Use the commutative and associative rules to simplify mental calculations and to check results.

Algebra and Functions (Grade Three)

1.5 Recognize and use the commutative and associative properties of multiplication (e.g., if $5 \times 7 = 35$, then what is 7×5? and if $5 \times 7 \times 3 = 105$, then what is $7 \times 3 \times 5$?).

Algebra and Functions (Grade Five)

1.3 Know and use the distributive property in equations and expressions with variables.

Students need to generalize these concepts through a wide range of examples showing the variation in the types of problems involving operations with whole numbers. For example, students should be exposed to decomposing numbers and simple "fact family" relationships in the inverse operations of addition and subtraction and of multiplication and division of whole numbers (e.g., $7 + 5 = 12$; $5 + 7 = 12$; $12 - 5 = 7$; $12 - 7 = 5$). Students who limit their use of the commutative, associative, and distributive rules for only two or three factors need to extend the rules to more factors.

These rules are the underpinnings of the algorithms used to perform arithmetic operations and are the basis for many proofs and reasoned arguments in mathematics. When students understand these rules, they can also justify the simplification of algebraic expressions and sequential, logical arguments in mathematics. The names of these rules should become part of a student's mathematics vocabulary, but more important, students should be able to use the rules appropriately. Students need to reach the point at which they can understand and express the rules, using symbolic notation; for example, for any numbers x, y, and z, it is always true by the associative rule that $x + (y + z) = (x + y) + z$. When a preamble such as "for any numbers x, y, and z" is expressed, students are introduced to the heart of algebra and the core of mathematical reasoning. This may be one of the first times when students have been exposed to the concept of generality, that a mathematical statement could be true "for any numbers" and not just "for some numbers."

IV-3. Evaluating Expressions

Algebra and Functions (Grade Seven)

1.2 Use the correct order of operations to evaluate algebraic expressions such as $3(2x + 5)^2$.

1.3 Simplify numerical expressions by applying properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

Without some knowledge of the conventions of writing mathematics, and the precise underlying concepts, students might be tempted to take an expression such as $3(2x + 5)^2$ and misapply the distributive rule to produce $(6x + 15)^2$. By knowing and using the order of operations, students will be less prone to make mistakes in solving and evaluating equations and inequalities.

IV-4. Equations and Inequalities

Algebra and Functions (Grade Four)

2.1 Know and understand that equals added to equals are equal.

2.2 Know and understand that equals multiplied by equals are equal.

Algebra and Functions (Grade Seven)

4.0 Students solve simple linear equations and inequalities over the rational numbers.

Students should come to a point where they can express an idea such as "equals added to equals are equal" symbolically (e.g., if a, b, x, and y are any four numbers, and $a = b$ and $x = y$, then $a + x = b + y$). The idea of maintaining equivalent expressions on both sides of the equal sign is fundamental to manipulating and simplifying equations. Students need to develop ways of manipulating equations by mathematical reasoning, changing a mathematical expression into an equivalent one, or correctly deriving a more useful expression from a given one.

IV-5. Symbolic Computation

Number Sense (Grade Seven)

1.3 Convert fractions to decimals and percents and use these representations in estimations, computations, and applications.

Students will have difficulty in working with equations and situations that use rational numbers if they have not gained proficiency with fractions, decimals, and percents. The student needs to be comfortable with simple problems of the type $\frac{A}{B} = \frac{?}{100}$, in which A and B are whole numbers. As far as the symbolic manipulation is concerned, it would not matter if A and B were any numbers (other than $B = 0$), but there is an added challenge if A and B are themselves fractions or decimals. The importance of complex fractions has already been discussed earlier (see III-3. "Rates").

Volume V Functions and Equations

This volume is about functions and equations and covers the following topics: V-1. Functions; V-2. Graphing; V-3. Proportional Relationships; and V-4. The Relationship Between Graphs and Functions.

V-1. Functions

Algebra and Functions (Kindergarten)

1.1 Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group (e.g., all these balls are green, those are red).

Statistics, Data Analysis, and Probability (Grade One)

1.1 Sort objects and data by common attributes and describe the categories.

2.1 Describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape).

Statistics, Data Analysis, and Probability (Grade Two)

2.1 Recognize, describe, and extend patterns and determine a next term in linear patterns (e.g., 4, 8, 12 . . . ; the number of ears on one horse, two horses, three horses, four horses).

Algebra and Functions (Grade Three)

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Algebra and Functions (Grade Four)

1.5 Understand that an equation such as y = 3x + 5 is a prescription for determining a second number when a first number is given.

Algebra and Functions (Grade Six)

2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

Students are gradually exposed to this topic over many years, but they may have difficulty if they do not possess some of the core concepts and skills at the appropriate times (e.g., an ability to count by 4s is likely to be a prerequisite to recognizing a pattern in a sequence 4, 8, 12, 16, 20, . . .).

At an introductory level, students need to understand what a function is, not in a formal sense, but in the sense that it is a rule that associates to each object of one kind an object of another kind. For example, the function $y = x^2$ yields the numbers 0, $\frac{1}{4}$, 1, 4, and 9 when x is 0, $\frac{1}{2}$, 1, 2, and –3, respectively. In the case of extending linear patterns, students need to develop and understand the explicit rule by which the pattern is extended. For example, given the sequence 2, 5, 8, 11, 14 . . . , students should recognize how to extend it as a linear pattern (add 3) and also determine the rule for generalizing this pattern (the nth number will be $3n - 1$). That knowledge will help students understand more advanced algebraic relationships.

V-2. Graphing

Statistics, Data Analysis, and Probability (Grade One)

1.2 Represent and compare data (e.g., largest, smallest, most often, least often) by using pictures, bar graphs, tally charts, and picture graphs.

Statistics, Data Analysis, and Probability (Grade Two)

1.1 Record numerical data in systematic ways, keeping track of what has been counted.

1.2 Represent the same data set in more than one way (e.g., bar graphs and charts with tallies).

Statistics, Data Analysis, and Probability (Grade Three)

1.3 Summarize and display the results of probability experiments in a clear and organized way (e.g., use a bar graph or a line plot).

Measurement and Geometry (Grade Four)

2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures.

2.1 Draw the points corresponding to linear relationships on graph paper (e.g., draw 10 points on the graph of the equation y = 3x and connect them by using a straight line).

Statistics, Data Analysis, and Probability (Grade Five)

1.4 Identify ordered pairs of data from a graph and interpret the meaning of the data in terms of the situation depicted by the graph.

1.5 Know how to write ordered pairs correctly; for example, (x, y).

A graph is the set of all the points that satisfy some condition. For example, if x and y are real numbers, the set of all the points (x, y) that satisfy the equation $y^2 = 1 - x^2$ is a circle (albeit not representing a function of x), and the set of all the points (x, y) that satisfy $y = 3x + 2$ is a straight line (representing a function of x). Students may have trouble in the transition from discrete graphs of integral values (e.g., total number of legs as a function of total number of horses) to graphs of a continuous relationship. And if students have not had much opportunity to engage in graphing practice, they may struggle to relate a point on the graph to the information it portrays, or they may make technical errors, such as reversing the x and y coordinates.

V-3. Proportional Relationships

Algebra and Functions (Grade Three)

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Number Sense (Grade Six)

1.3 Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7} = \frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

The types of difficulties students experience with ratios and rates may stem from the absence of a clear definition of what a ratio or a rate means and from any explanation of the concept of *constant rate.* For example, in the following problem: *If six workers can paint a barn in 3 days (working at the same constant* *rate), how many workers (working at the same constant rate) would be needed to paint the barn in 1 day?* Students may have difficulty in setting up proportions for such problems if they are unsure of which quantities are being compared, and they may not apply algorithms correctly if they have not been exposed to structured instruction that develops the appropriate prerequisite skills and concepts.

V-4. The Relationship Between Graphs and Functions

Algebra and Functions (Grade Five)

1.5 Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

Algebra and Functions (Grade Seven)

3.0 Students graph and interpret linear and some nonlinear functions.

3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.

3.3 Graph linear functions, noting that the vertical change (change in y-value) per unit of horizontal change (change in x-value) is always the same and know that the ratio ("rise over run") is called the slope of the graph.

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of the quantities.

Conceptual understanding of proportional relationships may be developed by using concrete examples and then moving toward a precise meaning for the ratio called "the slope." Students need to be aware that the slope of a line represents a constant rate and therefore can be computed using any two points on the line. This awareness can be developed by direct experimentation at this stage but will be justified in a later course in geometry. If the student does not have this solid foundation, graphing and its associated computations may be mystifying.
Apart from calculating the slope of a line, reading a graph requires that students interpret the scales and the axes. Students may be exposed exclusively to linear examples at an early age and be misled into thinking that all functions are linear so that all graphs of functions are straight lines. One way to correct this misconception is to expose them to many examples of functions whose graphs are not straight lines.

Volume VI Measurement

This volume is about measurement and covers the following topics: VI-1. How Measurements Are Made; VI-2. Length and Area in the Real World; VI-3. Exact Measure in Geometry; and VI-4. Angles and Circles.

VI-1. How Measurements Are Made

Algebra and Functions (Grade Three)

1.4 Express simple unit conversions in symbolic form (e.g., __ inches = __ feet × 12).

Measurement and Geometry (Grade Three)

1.4 Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).

Algebra and Functions (Grade Six)

2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

Measurement and Geometry (Grade Seven)

1.1 Compare weights, capacities, geometric measures, times, and temperatures within and between measurement systems (e.g., miles per hour and feet per second, cubic inches to cubic centimeters).

1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

Topic VI-1 is grounded in real-world applications of measurement. Students who do not have experience in making empirical measurements (e.g., knowing whether a gram or a kilogram is the greater mass) may have more difficulty in understanding conversions between systems of measurement.

VI-2. Length and Area in the Real World

Measurement and Geometry (Grade Two)

1.3 Measure the length of an object to the nearest inch and/or centimeter.

Measurement and Geometry (Grade Three)

1.2 Estimate or determine the area and volume of solid figures by covering them with squares or by counting the number of cubes that would fill them.

1.3 Find the perimeter of a polygon with integer sides.

Topic VI-2 introduces several new ideas that may need to be developed carefully through the use of concrete or semiconcrete examples. For example, measuring length to the nearest whole unit may be difficult if students start the measurement from "one" (one inch, one centimeter, or other one-linear unit) rather than from zero. Developing a sense of metric and standard U.S. measures may be challenging for some students.

VI-3. Exact Measure in Geometry

Measurement and Geometry (Grade Four)

1.1 Measure the area of rectangular shapes by using appropriate units, such as square centimeter (cm^2), square meter (m^2), square kilometer (km^2), square inch ($in.^2$), square yard ($yd.^2$), or square mile ($mi.^2$).

2.2 Understand that the length of a horizontal line segment equals the difference of the x-coordinates.

2.3 Understand that the length of a vertical line segment equals the difference of the y-coordinates.

Measurement and Geometry (Grade Five)

1.1 Derive and use the formula for the area of a triangle and of a parallelogram by comparing each with the formula for the area of a rectangle (i.e., two of the same triangles make a parallelogram with twice the area; a parallelogram is compared with a rectangle of the same area by pasting and cutting a right triangle on the parallelogram).

1.2 Construct a cube and rectangular box from two-dimensional patterns and use these patterns to compute the surface area for these objects.

1.3 Understand the concept of volume and use the appropriate units in common measuring systems (i.e., cubic centimeter [cm^3], cubic meter [m^3], cubic inch [$in.^3$], cubic yard [$yd.^3$]) to compute the volume of rectangular solids.

Topic VI-3 is more formal than the previous one as students move from concrete examples of measurements to understanding the formulas for computing area and volume. Students may have difficulty in shifting from linear measurements to square units and from additive operations to multiplicative operations. As such, the approximations and imperfections of concrete objects and computer models may not adequately represent the mathematical idea (e.g., a line drawn on paper has width, and a calculator provides "rounded" answers). The formulas for the area of a rectangle or triangle can readily be applied to the areas of faces of cubes and prisms, but students first need to understand that congruent figures have equal areas. Students may need practice in working with two-dimensional representations before they can generalize a formula for the total surface area of a three-dimensional figure. This topic of geometric measurement is critical because it sets the stage for geometry in later grades and the mathematical reasoning that underlies the subject. Students better understand linear, squared, and cubic units if they carry all units along in their calculations and use exponential notation for the units interchangeably with other abbreviations, as in $5 \text{ m} \times 2 \text{ m} = 10 \text{ m}^2$.

VI-4. Angles and Circles

Measurement and Geometry (Grade Five)

2.1 Measure, identify, and draw angles, perpendicular and parallel lines, rectangles, and triangles by using appropriate tools (e.g., straightedge, ruler, compass, protractor, drawing software).

2.2 Know that the sum of the angles of any triangle is 180° and the sum of the angles of any quadrilateral is 360° and use this information to solve problems.

Measurement and Geometry (Grade Six)

1.2 Know common estimates of π (3.14, $\frac{22}{7}$) and use these values to estimate and calculate the circumference and the area of circles; compare with actual measurements.

1.3 Know and use the formulas for the volume of triangular prisms and cylinders (area of base $\times$ height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

Measurement and Geometry (Grade Seven)

3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

3.4 Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.

Topic VI-4 develops many of the underlying ideas of geometry. Geometrical congruence, for example, is a concept that depends on reasoning. Students may have difficulty with these standards if their only experience has been memorizing formulas for the areas of triangles and parallelograms rather than developing and understanding the formulas through the use of mathematical reasoning. The formulas for areas and volumes of geometric figures and the Pythagorean theorem are sophisticated mathematical ideas that require a good grounding in prerequisite skills and concepts. Students need to understand that *pi* is the ratio between the circumference and diameter of all circles, the area of a unit circle, and an irrational number and that 3.14, or $\frac{22}{7}$, is only an approximation of pi. This topic develops some of the interesting and practical concepts of geometry and will be extended to a higher level in later grades.

Algebra Readiness Program
(Grade Eight or Above)

It is imperative for students, whether in grade eight, grade nine, or even a later grade, to master prealgebraic skills and concepts before they enroll in a course that meets or exceeds the rigor of the content standards for Algebra I adopted by the State Board of Education.

The 16 standards that are the target of the algebra readiness program specified in this section are organized into a set of nine topics (this organization is not required for the materials but is included for illustrative purposes). These 16 standards (13 from grade seven and three from Algebra I) are purposefully limited in number to provide publishers and teachers the flexibility and time to rebuild foundational skills and concepts that may be missing from earlier grades.

Nine Topics for the Algebra Readiness Program

Referenced Standards from Foundational Skills and Concepts	16 Targeted Standards for Algebra Readiness
Topic 1 Whole Numbers Grade 3—NS: 1.3, 1.5	**Grade 7**
Topic 2 Operations on Whole Numbers Grade 2—AF: 1.1 Grade 3—AF: 1.5 Grade 4—NS: 3.1, 3.2 Grade 5—AF: 1.3	**Topic 4 Operations on Rational Numbers** NS: 1.2, 1.3, 1.5, 2.1 AF: 2.1
Topic 3 Rational Numbers Grade 5—NS: 1.4 Grade 6—NS: 1.1	**Topic 6 Equations and Functions** AF: 1.1, 1.3, 4.1, 4.2
Topic 4 Operations on Rational Numbers Grade 6—NS: 1.4, 2.0, 2.1, 2.2	**Topic 7 The Coordinate Plane** MG: 3.3
Topic 5 Symbolic Notation Grade 4—AF: 1.2, 1.3 Grade 5—AF: 1.0 Grade 6—AF: 1.0, 1.1	**Topic 8 Graphing Proportional Relationships** AF: 3.3, 3.4 MG: 1.3
Topic 6 Equations and Functions Grade 4—AF: 1.5, 2.0, 2.1, 2.2	**Algebra I**
Topic 7 The Coordinate Plane Grade 4—MG: 2.0, 2.1, 2.2, 2.3 Grade 5—AF: 1.4	**Topic 9 Algebra** (Introductory Examples) Algebra I: 2.0, 4.0, 5.0

Abbreviations for the strands of the mathematics content standards are NS, Number Sense; AF, Algebra and Functions; MG, Measurement and Geometry; SDAP, Statistics, Data Analysis, and Probability; and MR, Mathematical Reasoning. **The standards from the Mathematical Reasoning strand are not displayed in this chart (see pp. 340–41 for the MR standards).**

These 16 standards define the subset of California mathematics standards that are the target of the algebra readiness program (see Chapter 10, criterion 13 in Category 1: "Mathematics Content/Alignment with the Standards"). The algebra readiness materials must also break these 16 standards into their component concepts and skills, with a primary focus on developing students' mastery of arithmetic. The checklists heading each section in the pages that follow provide guidance on the concepts and skills taught earlier that support instruction on the 16 standards. The instructional materials must provide diagnostic assessments on foundational concepts and skills and lessons that can be implemented in the classroom, as needed, to rebuild the missing foundational content. Additional guidance on assessment is provided in Chapter 5. It is crucial that materials for an algebra readiness program include large numbers of exercises and problems with a wide range of difficulty, starting with simple one-step problems and progressing to multistep problems for which students have become prepared. The program should be based on a set of highly focused instructional materials that break each standard into a series of small conceptual steps and embedded skills and should be designed to prepare students to complete a course in algebra successfully in the following year. Programs should provide support for a variety of instructional strategies, including various ways to explain and develop a concept.

Topic 1: Whole Numbers

Topic 1 Whole Numbers

Foundational skills and concepts

- ✓ **Concept of place value in whole numbers**
  (reference Grade 3 Number Sense 1.3)
- ✓ **Expanded form of whole numbers**
  (reference Grade 3 Number Sense 1.5)

Knowledge of the concepts and basic properties of numbers, and the ability to operate fluently with numbers, is an essential prerequisite for algebra. The discussion of whole numbers in "A Mathematics Intervention Program" (see I-2. "Place Value") is applicable here. For the purposes of algebra readiness, the students should work toward identification of digits in arbitrarily large numbers; however, as mentioned earlier, this concept may be difficult if students do not understand that adjacent columns (e.g., the thousands and hundreds places) are related by multiplication of ten. Students need to develop the ability to represent numbers by using exponents, such as $3{,}206 = 3 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$, once exponential notation has been introduced.

Topic 2 Operations on Whole Numbers

Foundational skills and concepts

- ✓ **Standard algorithms for addition and subtraction** (reference Grade 4 Number Sense 3.1)
- ✓ **Standard algorithms for multiplication and division** (reference Grade 4 Number Sense 3.2)
- ✓ **Associative and commutative rules** (reference Grade 2 Algebra and Functions 1.1 and Grade 3 Algebra and Functions 1.5)
- ✓ **Distributive rule** (reference Grade 5 Algebra and Functions 1.3)
- ✓ **Complete fluency with operations on whole numbers**

Issues of place value were discussed previously in "A Mathematics Intervention Program" (see I-3. "Addition and Subtraction"). Place value plays a critical role in all the arithmetic algorithms of whole numbers, as do the associative, commutative, and distributive rules, and students need to understand these connections.

For example, students using the standard long division algorithm to divide 345 by 7 would address the number 345 in place value columns, starting from the left.

$$\begin{array}{r|rrr} & & 4 & \\ \hline 7 & 3 & 4 & 5 \\ & 2 & 8 & \\ \hline & & 6 & 5 \end{array}$$

"Seven goes into 34 four times," they might say, but they must understand that 34 represents 34 tens, in the context of place value, and the 4×7 is correspondingly 40×7.

That is, $345 = (40 \times 7) + 65$, where 65 is the remainder from the first step. Long division is a series of successively closer approximations, and the information obtained thus far is that 345 lies between 40×7 and 50×7. Stated another way, $345 \div 7$ lies between 40 and 50. In the second step of a long division problem, students would refine their approximation and show that 345 lies between 49×7 and 50×7, meaning that $345 \div 7$ lies between 49 and 50. The exact solution is $345 = (49 \times 7) + 2$. This example is a standard form of writing a quotient and remainder. When students have the necessary foundation, they can use mathematical reasoning to show this process step by step:

$$\begin{aligned}
345 &= 340 + 5 && \text{(note place value)} \\
&= (40 \times 7) + 60 + 5 && \text{(arithmetic step)} \\
&= (40 \times 7) + 65 && \text{(associative property)} \\
&= (40 \times 7) + (9 \times 7) + 2 && \text{(arithmetic step)} \\
&= (40 + 9) \times 7 + 2 && \text{(distributive property)} \\
&= (49 \times 7) + 2 && \text{(arithmetic step)}
\end{aligned}$$

When students justify and understand each step in the long division algorithm, they gain crucial experience with the core types of mathematical reasoning essential for success in algebra.

The importance of the commutative, associative, and distributive rules, and consideration of their difficulty, has been discussed previously in "A Mathematics Intervention Program" (see IV-2. "Mathematical Fundamentals").

Algebra Readiness

Topic 3: Rational Numbers

Topic 3 Rational Numbers

Foundational skills and concepts

- ✓ **Definition of positive and negative fractions; number line representation**
- ✓ **Concept of a whole and its parts**
- ✓ **Concept of prime factorization and common denominators** (reference Grade 5 Number Sense 1.4)
- ✓ **Equivalence and ordering of positive and negative fractions** (reference Grade 6 Number Sense 1.1)
- ✓ **Expanded form of decimals using powers of ten**
- ✓ **Complete fluency with representing fractions, mixed numbers, decimals, and percentage**

Many of the difficulties students may have with rational numbers have been described previously in "A Mathematics Intervention Program" (see II-1. "Parts of a Whole"; II-2. "Equivalence of Fractions"; II-3. "Operations on Fractions").

Topic 4: Operations on Rational Numbers

Topic 4 Operations on Rational Numbers

Foundational skills and concepts

- ✓ **Definition of operations on fractions**
- ✓ **Mathematical reasoning with fractions in a structured, defined environment**
- ✓ **Understanding of why the standard algorithms work**
- ✓ **Complete fluency with operations on positive fractions** (reference Grade 6 Number Sense 1.4, 2.0, 2.1, and 2.2)

The following five standards are to be included in the instructional materials:

Number Sense (Grade Seven)

1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

1.3 Convert fractions to decimals and percents and use these representations in estimations, computations, and applications.

1.5 Know that every rational number is either a terminating or repeating decimal and be able to convert terminating decimals into reduced fractions.

2.1 Understand negative whole-number exponents. Multiply and divide expressions involving exponents with a common base.

Algebra and Functions (Grade Seven)

2.1 Interpret positive whole-number powers as repeated multiplication and negative whole-number powers as repeated division or multiplication by the multiplicative inverse. Simplify and evaluate expressions that include exponents.

Although simplifying fractions to their least common denominators is a useful skill, it is not as important at this stage as is understanding the equivalence of fractions. One important use of exponential notation is in the complete expanded form of a decimal (e.g., $32.61 = 3 \times 10^1 + 2 \times 10^0 + 6 \times 10^{-1} + 1 \times 10^{-2}$), which is important in building an understanding of place value and orders of magnitude and which mirrors what happens with polynomials (e.g., 3,216 is a special case of $3x^3 + 2x^2 + x + 6$, where x is replaced by 10).

Topic 5 Symbolic Notation

Foundational skills and concepts

- ✓ **Evaluating expressions with parentheses**
  (reference Grade 4 Algebra and Functions 1.2)
- ✓ **Writing equations using parentheses**
  (reference Grade 4 Algebra and Functions 1.3)
- ✓ **Using a "variable" to represent a number**
  (reference Grade 5 Algebra and Functions 1.0 and Grade 6 Algebra and Functions 1.1)
- ✓ **Complete fluency with the use of symbols to express verbal information**
  (reference Grade 6 Algebra and Functions 1.0)

This topic has been discussed in "A Mathematics Intervention Program" (see IV-3. "Evaluating Expressions").

Identifying the context of a word problem, picking a strategy for finding a solution, and expressing the solution in symbolic notation are all important components of algebraic problem solving. Students must be fluent in using symbols to express verbal information as a developmental step toward this goal. The topic of symbolic manipulation is developed here with the use of parentheses. Although the convention of order of operations becomes important in the writing of polynomials in symbolic notation, it is not critical mathematically at this point.

Topic 6 Equations and Functions

Foundational skills and concepts

✓ **The concept of an equation as a "prescription"** (reference Grade 4 Algebra and Functions 1.5)

✓ **The concept that equals added to equals are equal** (reference Grade 4 Algebra and Functions 2.1)

✓ **The concept that equals multiplied by equals are equal** (reference Grade 4 Algebra and Functions 2.2)

✓ **Basic techniques for manipulating symbols in an equation** (reference Grade 4 Algebra and Functions 2.0)

✓ **Complete fluency in writing and solving simple linear equations**

The following four standards are to be included in the instructional materials:

Algebra and Functions (Grade Seven)

1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).

1.3 Simplify numerical expressions by applying properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

4.1 Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.

4.2 Solve multistep problems involving rate, average speed, distance, and time or direct variation.

Understanding a functional relationship such as $y = 3x + 5$ may be significantly more difficult than understanding an equation such as $14 = 3x + 5$ because the nature of the information is different. In the first case the solution is an infinite set of ordered pairs, and in the second case, there is a unique solution.

Students may also have difficulty if they lack experience with concrete examples of functional relationships, such as rates, and with the various representations of these relationships (e.g., equation, table, graph, narrative).

Beginning algebra should be understood as generalized arithmetic. A letter such as x is used to represent only a number and nothing more. Computation with an expression in x is the same as ordinary calculations with specific, familiar numbers. In this way, beginning algebra becomes a natural extension of arithmetic. Symbols are also important for describing functions, but letters stand for numbers in this context as well. For example, in an expression such as $f(x) = 3x + 5$, the letter x represents a number, and $f(x)$ stands for the numerical value of the function when it is evaluated at the number x. Students will also need to recognize the expression $y = 3x + 5$ as another way to write $f(x) = 3x + 5$. Here the letter y stands for the number $f(x)$.

Topic 7 The Coordinate Plane

Foundational skills and concepts

- ✓ **Plotting and interpreting points (ordered pairs) on the coordinate plane**
  (reference Grade 4 Measurement and Geometry 2.0 and Grade 5 Algebra and Functions 1.4)
- ✓ **Plotting lines and simple polygons based on a "recipe" or set of instructions**
- ✓ **Graphing lines corresponding to simple linear equations, as a "prescription"**
  (reference Grade 4 Measurement and Geometry 2.1)
- ✓ **The concept that a graph is a collection of *all* the ordered pairs satisfying a defined condition**
- ✓ **Complete fluency in plotting points, interpreting ordered pairs from a graph, and interpreting lengths of horizontal and vertical line segments on a coordinate plane**
  (reference Grade 4 Measurement and Geometry 2.2 and 2.3)

The following standard is to be included in the instructional materials:

Measurement and Geometry (Grade Seven)

3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

Many of the difficulties students may have with graphing have been described previously in "A Mathematics Intervention Program" (see V-2. "Graphing" and V-4. "The Relationship Between Graphs and Functions"). The skills of graphing, and the related use of the Pythagorean theorem in a geometric distance formula, are key elements in the later study of algebra.

Topic 8 Graphing Proportional Relationships

Foundational skills and concepts

- ✓ **Ratio and proportion; drawing and reading graphs of lines passing through the origin**
- ✓ **The geometric context for ratio and proportion; similar right triangles on a graph**
- ✓ **The concept of the slope of a line**
- ✓ **Complete fluency in graphing and interpreting relationships of the form $y = mx$**

The following three standards are to be included in the instructional materials:

Algebra and Functions (Grade Seven)

3.3 Graph linear functions, noting that the vertical change (change in y-value) per unit of horizontal change (change in x-value) is always the same and know that the ratio ("rise over run") is called the slope of a graph.

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of the quantities.

Measurement and Geometry (Grade Seven)

1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

Graphs are important tools for representing functional relationships in algebra, and they visually reinforce difficult concepts, such as ratio and proportionality.

Topic 9 Algebra (Introductory Examples)

The following three standards are to be included in the instructional materials:

Algebra I (Grades Eight Through Twelve)

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents [excluding fractional powers].

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as 3(2x – 5) + 4(x – 20) = 12 [excluding inequalities].

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step [excluding inequalities].

These algebra standards are to be covered at the end of an algebra readiness course. Teachers may use this information, as appropriate, to introduce Algebra I concepts to students at the end of the school year (approximately two to four weeks of instructional time is suggested). Students may have difficulty with these introductory examples if their grasp of the algebra readiness content has not been thorough or if the examples are presented at too challenging a level. If students do not understand the relationship between a stated problem, a related symbolic equation, and a graphical representation (the set of points satisfying the equation), they may find algebraic discussions difficult. Simplifying an expression such as

$$\frac{2\left(x^2-1\right)}{\left(x+1\right)}$$

may be challenging if students do not have enough experience in strategic use of the distributive rule or with multistep problems. Understanding symbolic equations and inverse operations is key to algebraic problem solving. Through the study of algebra, students will see the usefulness of equations and learn the appropriate mechanics for manipulating them.